GANTMACHER EXAMPLE

1. The Matrix

In Theory of Matrices, vol. II, Gantmacher constructs a nilpotent complex symmetric matrix, $C = A + iB$, where A and B are zero except

$$a_{k,k+1} = a_{k+1,k} = 1, \tag{1}$$
$$b_{n-k,k} = -b_{n-k+1,k+1} = 1, \tag{2}$$

for $k = 1, \ldots, n-1$.

1.1. The Almost Principal Vectors.

$$x_k = e_k - ie_{n-k+1}., \tag{3}$$

for $k = 1, \ldots, n$. Note that

$$\begin{aligned} x_{n-k+1} &= e_{n-k+1} - ie_{n-(n-k+1)+1} =, e_{n-k+1} - ie_k \\ &= i\bar{x}_k \end{aligned} \tag{4}$$

Although the x_k are not orthogonal,

$$\bar{x}_k^T x_j = 2(\delta_{kj} - i\delta_{k,n-j+1}). \tag{5}$$

However, they are almost principal vectors since $Cx_1 = 0$, and for $k = 2, \ldots, n-1$

$$\begin{aligned} Cx_k &= Ce_k - iCe_{n-k+1}, \\ &= Ae_k + iBe_k - i(Ae_{n-k+1} + iBe_{n-k+1}), \\ &= Ae_k + iBe_k - iAe_{n-k+1} + Be_{n-k+1}), \\ &= e_{k-1} + e_{k+1} + iBe_k - i(e_{n-k} + e_{n-k+2}) + Be_{n-k+1}, \\ &= e_{k-1} + e_{k+1} + i(e_{n-k} - e_{n-k+2}) - i(e_{n-k} + e_{n-k+2}) + e_{k-1} - e_{k+1}, \\ &= 2e_{k-1} + i(e_{n-k} - e_{n-k+2}) - i(e_{n-k-1} + e_{n-k+1}), \\ &= 2(e_{k-1} + ie_{n-k+2}) = 2x_{k-1}, \end{aligned} \tag{6}$$

1.2. The Field of Values.

Define X as the matrix whose columns are $(x_1, \ldots, x_n)$, and $z = Xy$,

$$\begin{aligned} z^*Cz &= y^*X^TCXy \\ &= y^*X^TCXy \\ &= y^*Hy, \end{aligned} \tag{7}$$

(8)

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where $H = X^T C X$, the first column of H is zero, and from (5)

$$\begin{aligned} h_{kj} &= x_k^* x_{j-1}, \\ &= \bar{x}_k x_{j-1} = 2(\delta_{k,j-1} - i\delta_{k,n-j+2}) \end{aligned} \tag{9}$$